Math 146C - Ordinary and Partial Differential Equations III
Quiz 5
May 19, 2011

Name: Key

| Total |
|---|
| /10 |

**Problem 1.** *(10 points) Consider the boundary value problem*

$$\begin{cases} y'' + \lambda y = 0 \\ y'(0) = y'(1) = 0 \end{cases}$$

*Find the normalized eigenfuctions of the problem and show that they form an orthonormal family.*

$\underline{\lambda<0}$: $\lambda = -\mu^2 \Rightarrow y = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$y' = \mu c_1 e^{\mu x} - \mu c_2 e^{-\mu x}$

$y'(0) = \mu(c_1 - c_2) = 0 \Rightarrow c_1 = c_2 \quad (\mu \neq 0)$

$y'(1) = \mu c_1 (e^{\mu} - e^{-\mu}) = 0 \Rightarrow c_1 = c_2 = 0 \quad \left(\begin{matrix}\mu \neq 0 \\ e^{\mu} \neq e^{-\mu} \text{ (e is 1-1)}\end{matrix}\right)$

No negative e.vals.

---

$\underline{\lambda=0}$: $\lambda = 0 \Rightarrow y = c_1 x + c_2 \Rightarrow y' = c_1$

$y'(0) = c_1 = 0 \Rightarrow y = c_2$. So for the eigenvalue $\lambda=0$, the eigenfunction is $y_0 = 1$.

---

$\underline{\lambda>0}$: $\lambda = \mu^2 \Rightarrow y = c_1 \cos \mu x + c_2 \sin \mu x$, $y' = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$

$y'(0) = \mu c_2 = 0 \Rightarrow c_2 = 0$; $y'(1) = -\mu c_1 \sin \mu$. For nontrivial sol, $\mu = n\pi$

$\Rightarrow \lambda_n = n^2\pi^2$ & $y_n = \cos n\pi x$, $n \in \mathbb{N}$.

To normalize we need $k_n$ s.t. $\int_0^1 (k_n y_n)^2 dx = 1$, then $\phi_n = k_n y_n$.

<u>n=0</u>: $\int_0^1 (k_0 \cdot y_0)^2 dx = \int_0^1 k_0^2 dx = k_0^2 = 1 \Rightarrow k_0 = 1 \quad \therefore \phi_0 = 1$

<u>n≥1</u>: $\int_0^1 (k_n \cdot y_n)^2 dx = \int_0^1 k_n^2 \cos^2 n\pi x \, dx = k_n^2 \int_0^1 \frac{1+\cos 2n\pi x}{2} dx$

$$= k_n^2 \left( \frac{1}{2}x + \frac{1}{4n\pi} \sin 2n\pi x \right) \Big|_0^1$$

$$= k_n^2 \left( \left[\frac{1}{2} + 0\right] - [0+0] \right) = \frac{k_n^2}{2} = 1 \Rightarrow k_n^2 = 2 \Rightarrow k_n = \sqrt{2}$$

$\therefore \phi_n = \sqrt{2} \cos n\pi x$

---

Let $n, m \in \mathbb{N}$, $n \neq m$.

$$\int_0^1 \phi_0 \phi_n \, dx = \int_0^1 \sqrt{2} \cos n\pi x \, dx = \frac{\sqrt{2}}{n\pi} \sin n\pi x \Big|_0^1 = 0 - 0$$

$\therefore \phi_0$ & $\phi_n$ are orthogonal $\forall n \in \mathbb{N}$.

$$\int_0^1 \phi_n \phi_m \, dx = 2\int_0^1 \cos n\pi x \cos m\pi x \, dx = \cancel{2}\int_0^1 \cancel{\tfrac{1}{2}}[\cos(n+m)\pi x + \cos(n-m)\pi x] \, dx$$

$$= \left( \frac{\sin(n+m)\pi x}{n+m} + \frac{\sin(n-m)\pi x}{n-m} \right) \Big|_0^1 = 0$$

$\therefore \phi_n$ is orthogonal to $\phi_m$ for $n \neq m$; $n, m \in \mathbb{N}$.

$\therefore \{\phi_n\}_{n=0}^{\infty}$ is an orthonormal family.